Geometric monodromy around the tropical limit

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1. Introduction

Let $\{V_q\}_q$ be a complex one-parameter family of complex hypersurfaces. We give a concrete description of the monodromy transformation of $\{V_q\}_q$ around $q = \infty$ in terms of tropical geometry. The motivation comes from the calculation of monodromies of period maps.

2. Setting

- $K := \mathbb{C}\{t\}$: the convergent Laurent series field
- $f = \sum_{i \in A} k_i x^i \in K\left[x_1^{\pm}, \dots, x_{n+1}^{\pm}\right]$ $\quad \left(A \subset \mathbb{Z}^{n+1}\ :\ \text{a finite subset}\right)$
- Fix $R \gg 1$
- For each $q \in S_R^1 := \{z \in \mathbb{C} \mid |z| = R\}$, we set

$$f_q := f|_{t=1/q} \in \mathbb{C}\left[x_1^{\pm}, \dots, x_{n+1}^{\pm}\right]$$

- V_q : the complex hypersurface defined by f_q

We describe the monodromy of $\{V_q\}_{q \in S_R^1}$ in terms of tropical geometry.

3. Tropical Geometry

Tropical geometry : Algebraic geometry over the tropical semi-ring $(\mathbb{T} := \mathbb{R} \cup \{-\infty\}, \oplus, \odot)$

$$a \oplus b := \max\{a, b\},$$
$$a \odot b := a + b.$$

Tropical polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$

$$F(X_1, \dots, X_{n+1}) := \oplus_{i \in A} \left(a_i \odot X_1^{i_1} \odot X_2^{i_2} \odot \cdots \odot X_{n+1}^{i_{n+1}}\right)$$
$$= \max_{i \in A} \{a_i + i_1 \cdot X_1 + i_2 \cdot X_2 + \cdots + i_{n+1} \cdot X_{n+1}\},$$

where $A \subset \mathbb{Z}^{n+1}$ is a finite subset and $a_i \in \mathbb{R}$.

Tropical hypersurface $V(F) \subset \mathbb{R}^{n+1}$ defined by F

$$V(F) := \left\{X \in \mathbb{R}^{n+1} \mid F \text{ is not differentiable at } X\right\}$$

$v : K^* \to \mathbb{Z}$: the standard non-Archimedean valuation of K

$$v\left(\sum_{j \in \mathbb{Z}} c_j t^j\right) = -\min\left\{j \in \mathbb{Z} \mid c_j \neq 0\right\}$$

Tropicalization

$$\begin{array}{ccc} K\left[x_1^{\pm}, \cdots, x_{n+1}^{\pm}\right] & & \text{tropical polynomial} \\ \text{\rotatebox{90}{$\in$}} & \rightsquigarrow & \\ f = \sum_{i \in A} k_i x^i & & \operatorname{trop}(f) \end{array}$$

$$\operatorname{trop}(f)(X_1, \cdots, X_{n+1}) := \max_{i \in A} \{v(k_i) + i_1 \cdot X_1 + \cdots + i_{n+1} \cdot X_{n+1}\}$$

4. Monodromy in the case $n = 1$

Assume $n = 1$ and $V(\operatorname{trop}(f))$ is smooth.

- $\{\rho_i\}_{i=1,\cdots,d}$: the set of all bounded edges of $V(\operatorname{trop}(f))$
- L_i : the length of ρ_i
- C_i : the simple closed curve on V_q corresponding to ρ_i
- $T_i : V_{q=R} \to V_{q=R}$: the Dehn twist along C_i

Theorem [3]

The monodromy transformation of $\{V_q\}_{q \in S_R^1}$ is given by $T_1^{L_1} \circ \cdots \circ T_d^{L_d}$.

This is conjectured by Iwao [1]. A concrete description of the monodromy of $\{V_q\}_{q \in S_R^1}$ in any dimension n is also given in [3].

5. Example

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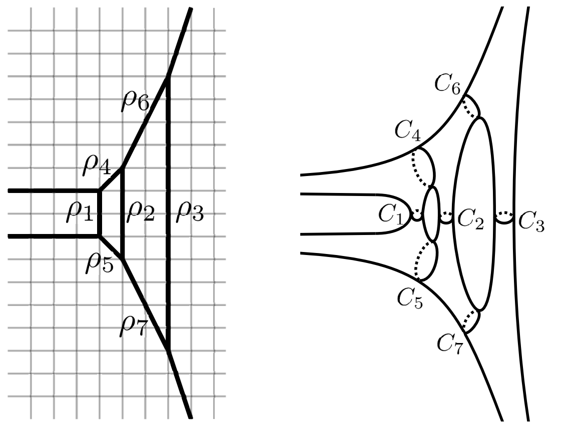
$$f(x_1, x_2) = x_2^2 + x_2\left(x_1^3 + t^{-2}x_1^2 + t^{-2}x_1 + t^{-1}\right) + 1$$

$$f_q(x_1, x_2) = x_2^2 + x_2\left(x_1^3 + q^2x_1^2 + q^2x_1 + q\right) + 1$$
$$\operatorname{trop}(f)(X_1, X_2) = \max\{2X_2, 3X_1 + X_2, 2X_1 + X_2 + 2, X_1 + X_2 + 2, X_2 + 1, 0\}$$



Tropical hypersurface $V(\operatorname{trop}(f))$ and complex hypersurface V_q

- Lengths of edges

$$L_1 = 2, \quad L_2 = 4, \quad L_3 = 12, \quad L_4 = L_5 = 1, \quad L_6 = L_7 = 2$$

- Monodromy transformation

$$T_1^2 \circ T_2^4 \circ T_3^{12} \circ T_4 \circ T_5 \circ T_6^2 \circ T_7^2$$

6. Idea of the proof

1. Deform V_q isotopically to a simpler manifold W_q by neglecting lower order terms (e.g. $f_q \approx qx_2 + q^2x_1x_2$ around ρ_1 on the above example). This method is introduced by Mikhalkin [2].
2. The monodromy transformation of $\{W_q\}_{q \in S_R^1}$ is easy to describe.

References

[1] Shinsuke Iwao, Lecture at the Mathematical Society of Japan Autum Meeting 2010, Video is available at http://mathsoc.jp/videos/2010shuuki.html, 2010.

[2] Grigory Mikhalkin, Decomposition into pairs-of-pants for complex algebraic hypersurfaces, Topology 43 (2004), no. 5, 1035-1065.

[3] Yuto Yamamoto, Geometric monodromy around the tropical limit, arXiv:1509.00175.